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Free convection in a shallow cavity with variable properties—1. Newtonian fluid

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Abstract—The effect of variable properties on free convection in a shallow cavity with differentially heated end walls has been analytically studied. Compared to the results for the *linear* theory the Boussinesq approximation usually applied is exact, if the fluid properties are taken at the arithmetic mean between the hot and cold temperature at the end walls. For the parameter range considered the deviation between the results for the *quadratic theory* and the Boussinesq approximation is less than approximately 2.5%. Hence, it has been demonstrated that the arithmetic mean is a reasonable choice. With this reference temperature the Boussinesq approximation leads to sufficiently accurate results.

1. INTRODUCTION

IN THEORETICAL studies on free convection problems the Boussinesq approximation is usually applied; i.e. all properties are taken as constant except the density in the buoyancy term in the equation of motion.

For external flows, some work has been carried out on the effect of variable properties on the momentum and heat transport and, hence, on the deviation from the Boussinesq approximation. Free convection heat transfer with variable properties on a *vertical plate* has been studied by Sparrow and Gregg [1], Fujii *et al.* [2], Miyamoto [3], Carey and Mollendorf [4, 5], Herwig *et al.* [6], and Herwig [7]. From these studies it follows that the effect of variable properties can be taken into account either by applying the so-called *reference-temperature concept*, i.e. taking the properties at

$$T_R = T_\infty + v(T_W - T_\infty) \quad (1)$$

and calculating the heat transfer from the Boussinesq solution (i.e. for constant properties) or by applying the *property-ratio concept*

$$\frac{Nu}{Nu_B} = (\rho^*\eta^*)^{n_1}\left(\frac{\rho^*}{\beta^*}\right)^{n_2}(\rho^*\lambda^*)^{n_3}(c_p^*)^{n_4} \quad (2)$$

where Nu_B is the Nusselt number from the Boussinesq solution. The constant v in equation (1) can be any value between 0 and 1. The exponents n_i in equation (2) are in general functions of the properties themselves, especially of the Prandtl number. These coefficients can be determined by analytical or numerical solution of the governing equation, and, in principle, also by carrying out an experimental study.

In opposition of these studies, very little work has been done on the effect of variable properties on free convection *in cavities*. Gray and Giorgini [8] have considered the effect of variable properties on the onset of convection in a horizontal fluid layer heated from below (Rayleigh–Bénard problem) by estimating the order of magnitude of each term appearing in the governing equations. Yamasaki and Irvine [9] studied the effect of variable viscosity on the free convection in a heated vertical pipe by solving the governing equation numerically. Their results show that the heat transfer rate is considerably improved when taking the variable viscosity into account.

Free convection in a *shallow cavity* with differently heated end walls and constant properties has been widely studied by Cormack *et al.* [10, 11], Imberger [12] and Bejan and Tien [13]. Accordingly, besides the analytical and numerical results also experimental data are known for this particular geometry. As an analytical study has the essential advantage of permitting a 'deeper look' into the physical and mathematical context of the problem, we have chosen this particular geometry for our study. Finally, we have carried out an analytical solution for free convection with variable properties using the method of asymptotic expansion. By comparing our results with those mentioned above, we discuss the effect of variable properties and, hence, deviations from the Boussinesq approximation.

2. MATHEMATICAL FORMULATION

Figure 1 shows a schematic sketch of the shallow cavity. The end walls are held at constant but different temperatures, the lower and the upper wall are adiabatic. Due to the horizontal temperature difference and uni-cellular flow is induced; the resulting velocity and temperature profile in the core region is also shown in Fig. 1.

Energy and momentum transport in the fluid are described by the equation of continuity

$$A\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0 \quad (3)$$

the equations of motion

NOMENCLATURE

a thermal diffusivity [$m^2 s^{-1}$]
c_p heat capacity J $kg^{-1} K^{-1}$
g gravitational acceleration constant [$m s^{-2}$]
h cavity height [m]
l cavity length [m]
n_i exponents in equations (2) and (37) [—]
p pressure [$N m^{-2}$]
T temperature [K]
u, v horizontal and vertical velocity components [$m s^{-1}$]
x, y horizontal and vertical coordinates [m].

Greek symbols
β coefficient in thermal expansion [K^{-1}]
ν factor in equation (1) [—]
ε non-dimensional temperature (perturbation parameter) [—]
η dynamic viscosity [Pa s]
θ dimensionless temperature [—]
λ thermal conductivity [$W K^{-1} m^{-1}$]
ν kinematic viscosity [$m^2 s^{-1}$]
ρ density [$kg m^{-3}$]
ψ stream function [—].

Dimensionless groups
A cavity aspect ratio
Gr Grashof number
K_{α_i} dimensionless property of the second kind
Nu Nusselt number
Pr Prandtl number.

Subscripts
B Boussinesq approximation
C cold
H hot
R reference state.

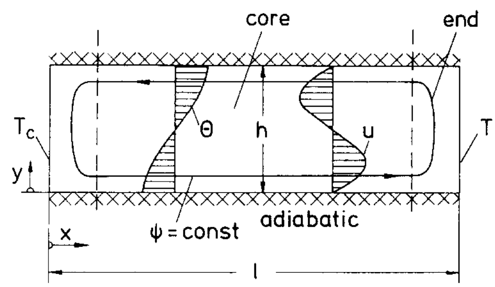

FIG. 1. Schematic sketch of the shallow cavity with velocity and temperature profiles in the core region.

$$Gr\,\rho\left[A^2 u\frac{\partial u}{\partial x}+Av\frac{\partial u}{\partial y}\right] = -Gr\frac{\partial p}{\partial x}+A\frac{\partial}{\partial x}\left[2A\eta\frac{\partial u}{\partial x}-\frac{2}{3}\eta\,\mathrm{div}\,\mathbf{v}\right] + \frac{\partial}{\partial y}\left[\eta\frac{\partial u}{\partial y}+A\eta\frac{\partial v}{\partial x}\right] \quad (4)$$

$$Gr\,\rho\left[A^3 u\frac{\partial v}{\partial x}+A^2 v\frac{\partial v}{\partial y}\right] = -Gr\frac{\partial p}{\partial y}+A\frac{\partial}{\partial y}\left[2\eta\frac{\partial v}{\partial y}-\frac{2}{3}\eta\,\mathrm{div}\,\mathbf{v}\right] + A^2\frac{\partial}{\partial x}\left[\eta\frac{\partial u}{\partial y}+A\eta\frac{\partial v}{\partial x}\right]+\frac{\rho_R h^2 g}{\eta_R u_R}(1-\rho) \quad (5)$$

and the equation of thermal energy

$$Gr\,Pr\,\rho c_p\left[A^2 u\frac{\partial\theta}{\partial x}+Av\frac{\partial\theta}{\partial y}\right] = A^2\frac{\partial}{\partial x}\left\{\lambda\frac{\partial\theta}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\lambda\frac{\partial\theta}{\partial y}\right\} \quad (6)$$

as well as by the boundary conditions

$$\begin{aligned} x=0:&\quad u=v=0, &\theta=\theta_C\\ x=1:&\quad u=v=0, &\theta=\theta_H\\ y=0,1:&\quad u=v=\frac{\partial\theta}{\partial y}=0. \end{aligned} \quad (7)$$

Equations (3)–(6) have already been made dimensionless with the reference numbers given in Table 1. The properties have been related to those taken at a certain reference temperature. As dimensionless groups show up the Grashof number, $Gr = \rho_R u_R h/\eta_R$, the Prandtl number, $Pr = \nu_R/a_R$, and the cavity aspect ratio $A = h/L$. The properties appearing in the Grashof and Prandtl numbers are taken at a certain reference temperature T_R.

The temperature dependence of the variable properties are taken into account by expanding them into a Taylor series fixed at a certain reference state.

Neglecting terms with order higher than two, one obtains, e.g. for the density

$$\rho^*(T) = \rho_R+\left(\frac{\partial\rho}{\partial T}\right)_R(T-T_R)+\frac{1}{2}\left(\frac{\partial^2\rho}{\partial T^2}\right)_R(T-T_R)^2. \quad (8)$$

Defining dimensionless properties of the 'second kind'

Table 1. Dimensionless variables and properties

x	y	u	v	p	θ	ρ	η	λ	c_p
$\frac{x^*}{l}$	$\frac{y^*}{h}$	$\frac{u^*}{Au_R}$	$\frac{v^*}{Au_R}$	$\frac{p^*}{\rho_R u_R^2}$	$\frac{T-T_R}{T_H-T_K}$	$\frac{\rho^*}{\rho_R}$	$\frac{\eta^*}{\eta_R}$	$\frac{\lambda^*}{\lambda_R}$	$\frac{c_p^*}{c_{pR}}$

Table 2. Dimensionless properties of the second kind for water and air at 70°C

	K_{ρ_1}	K_{η_1}	K_{λ_1}	$K_{c_{p2}}$	K_{ρ_2}	K_{η_2}	K_{λ_2}	$K_{c_{p2}}$
Water	−0.205	−4.749	0.493	0.045	−0.609	38.61	−1.981	0.262
Air	−1.0	0.722	0.882	−0.187	2.0	−0.382	−0.382	1.430

$$K_{\rho_1} = \left(\frac{T}{\rho}\frac{\partial \rho}{\partial T}\right)_R = -\beta_R T_R \tag{9}$$

$$K_{\rho_2} = \left(\frac{T^2}{\rho}\frac{\partial^2 \rho}{\partial T^2}\right)_R \tag{10}$$

and an expansion parameter

$$\varepsilon = \frac{T_H - T_C}{T_R} \tag{11}$$

one obtains from equation (8)

$$\rho = 1 + \varepsilon K_{\rho_1}\theta + \frac{\varepsilon^2}{2} K_{\rho_2}\theta^2. \tag{12}$$

Similar to this, one obtains for the other properties

$$\eta = 1 + \varepsilon K_{\eta_1}\theta + \frac{\varepsilon^2}{2} K_{\eta_2}\theta^2 \tag{13}$$

$$\lambda = 1 + \varepsilon K_{\lambda_1}\theta + \frac{\varepsilon^2}{2} K_{\lambda_2}\theta^2 \tag{14}$$

$$c_p = 1 + \varepsilon K_{c_{p1}}\theta + \frac{\varepsilon^2}{2} K_{c_{p2}}\theta^2. \tag{15}$$

Table 2 shows values for the properties of the second kind for water and air at $T_R = 70°C$ according to Gersten and Herwig [14]. In the present study we have chosen as reference temperature the arithmetic mean between the end wall temperatures

$$T_R = \frac{T_C + T_H}{2} \tag{16a}$$

as well as the temperature of the cold end wall

$$T_R = T_C. \tag{16b}$$

As there is no characteristic reference velocity u_R in free convection problems, we look at the buoyancy term. Substituting equation (12) into equation (5) leads to

$$\frac{\rho_R h^2 g}{\eta_R U_R}(1-\rho) = -\frac{\rho_R h^2 g}{\eta_R U_R}\left(\varepsilon K_{\rho_1}\theta + \frac{\varepsilon^2}{2} K_{\rho_2}\theta^2\right). \tag{17}$$

This relation shows that the buoyancy term is of $O(\varepsilon)$, whereas all other terms in equation (5) are of $O(1)$. Since the buoyancy term acts as generator for free convection, the reference velocity u_R has to be chosen in such a way that the buoyancy term becomes of $O(1)$ as $\varepsilon \to 0$. This demand yields

$$u_R = -\frac{\rho_R h^2 g \varepsilon K_{\rho_1}}{\eta_R}. \tag{18}$$

Substituting this reference velocity into the Grashof number gives

$$Gr = -\frac{\rho_R^2 h^3 g \varepsilon K_{\rho_1}}{\eta_R^2}. \tag{19}$$

Furthermore, considering

$$\varepsilon K_{\rho_1} = -\beta_R(T_H - T_C) \tag{20}$$

and

$$\nu_R = \frac{\eta_R}{\rho_R} \tag{21}$$

one obtains finally the relation for the Grashof number

$$Gr = \frac{g h^3 \beta_R (T_H - T_C)}{\nu_R^2} \tag{22}$$

which is identical with the form usually used in free convection problems.

Similar to the properties, the unknown quantities u, v, p and θ were also expanded in asymptotic expressions, using ε as a parameter. Hence

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \tag{23}$$

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots \tag{24}$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \tag{25}$$

$$\theta = \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \cdots. \tag{26}$$

In the following we talk of *linear theory*, if expansions (23)–(26) are broken off according to terms of $O(\varepsilon^1)$ and of *quadratic theory*, if they are broken off according to $O(\varepsilon^2)$.

Substituting the property expansions (12)–(15) as well as expansions (23)–(26) in the governing equations and equating terms of like power in ε, one obtains at $O(\varepsilon^0)$

$$A\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}=0 \tag{27}$$

$$Gr\left[A^2u_0\frac{\partial u_0}{\partial x}+Av_0\frac{\partial u_0}{\partial y}\right] = -Gr\frac{\partial p_0}{\partial x}+A^2\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2} \tag{28}$$

$$Gr\left[A^3u_0\frac{\partial v_0}{\partial x}+A^2v_0\frac{\partial v_0}{\partial y}\right] = -Gr\frac{\partial p_0}{\partial y}+A^3\frac{\partial^2 v_0}{\partial x^2}+A\frac{\partial^2 v_0}{\partial y^2}+\theta_0 \tag{29}$$

$$Gr\,Pr\left[A^2u_0\frac{\partial \theta_0}{\partial x}+Av\frac{\partial \theta_0}{\partial y}\right]=A^2\frac{\partial^2\theta_0}{\partial x^2}+\frac{\partial^2\theta_0}{\partial y^2}. \tag{30}$$

It is easily seen that this system of differential equations is identical with that resulting from the Boussinesq approximation. The system at $O(\varepsilon^1)$ is given in the Appendix.

With the assumption that the cavity aspect ratio A is sufficiently small, the resulting systems of differential equations at $O(\varepsilon^0)$, $O(\varepsilon^1)$ and $O(\varepsilon^2)$ can be solved using the asymptotic expansion method with the cavity aspect ratio A as the expansion parameter.

3. ANALYTICAL SOLUTION

For two-dimensional problems one usually introduces the stream function

$$\rho u = \frac{\partial \psi}{\partial y} \tag{31a}$$

$$\rho v = -A\frac{\partial \psi}{\partial x} \tag{31b}$$

which identically fulfills the continuity equation at each order of magnitude. By eliminating the pressure in equations (28) and (29), one obtains the vorticity equation.

As already mentioned, one distinguishes between two different flow regions the core region and the two end regions close to the hot and cold end walls. For $A \to 0$ equations (27)–(30) describe the flow in the core region. By solving these equations with the method of matched asymptotic expansions, some constants appear in the solution which must be determined. This is usually done by solving the equations for the end region and matching this solution with that for the core region. The equations valid in the end region are obtained from equations (27)–(30) by stretching the x-coordinate by the factor A^{-1}, see Cormack *et al.* [10]. Cormack *et al.* solved the equations for the end region numerically. However, Merker and Leal [15] have shown that the core solution correct to $O(A^2)$ can be obtained without knowing the end solution in detail.

Finally, one obtains with $T_R = (T_C+T_H)/2$ for the velocity $u(x,y)$

$$u=\frac{4y^3-6y^2+2y}{24}\Bigg[1+\varepsilon\left(x-\frac{1}{2}\right)\left(\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-K_{\lambda_1}\right)$$
$$+\varepsilon^2\Bigg\{\left(x-\frac{1}{2}\right)^2\left(K_{\eta_1}^2+K_{\lambda_1}K_{\eta_1}-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}\right.$$
$$\left.+\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\right)+\frac{x}{2}(x-1)\left(\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-K_{\eta_1}K_{\lambda_1}\right)$$
$$+\frac{1}{2}\left(x^2-x+\frac{1}{6}\right)(3K_{\lambda_1}^2-K_{\lambda_2})\Bigg\}\Bigg] \tag{32}$$

and for the temperature θ in the core region

$$\theta = x-\frac{1}{2}+\varepsilon\left[\frac{K_{\lambda_1}}{2}(x-x^2)\right]$$
$$+\varepsilon^2\left[(3K_{\lambda_1}^2-K_{\lambda_2})\left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12}\right)\right]$$
$$+A^2\frac{Gr\,Pr}{24}\left(\frac{y^5}{5}-\frac{y^4}{2}+\frac{y^3}{3}-\frac{1}{60}\right)$$
$$+\varepsilon A^2\left\{\frac{Gr\,Pr}{24}\left(\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3-\frac{1}{60}\right)\left(x-\frac{1}{2}\right)\right.$$
$$\times\left(K_{\rho_1}+\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-3K_{\lambda_1}+K_{c_{p1}}\right)-\frac{Gr^2\,Pr^2}{362\,880}$$
$$\times\left[2K_{\rho_1}+2\frac{K_{\rho_2}}{K_{\rho_1}}-2K_{\eta_1}-4K_{\lambda_1}+3K_{c_{p1}}\right]$$
$$\left.\times\left(\frac{x^2}{2}-\frac{x}{2}\right)\right\}+\varepsilon^2A^2\left\{\frac{Gr\,Pr}{24}\left(\frac{1}{5}y^5-\frac{1}{2}y^4+\frac{1}{3}y^3\right.\right.$$
$$\left.-\frac{1}{60}\right)\left[\left(x-\frac{1}{2}\right)^2\left(K_{\eta_1}^2+3K_{\eta_1}K_{\lambda_1}-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}\right.\right.$$
$$-K_{\rho_1}K_{\eta_1}-\frac{K_{\eta_2}}{2}-3\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-3K_{\lambda_1}K_{\rho_1}-\frac{3}{2}K_{\rho_2}$$
$$+4K_{\lambda_1}^2+K_{c_{p1}}\frac{K_{\rho_2}}{K_{\rho_1}}+K_{c_{p1}}K_{\rho_1}-K_{c_{p1}}K_{\eta_1}$$
$$\left.-3K_{c_{p1}}K_{\eta_1}+\frac{K_{c_{p2}}}{2}-\frac{K_{\lambda_2}}{2}\right)+(x-x^2)$$
$$\times\left(\frac{K_{\rho_2}}{K_{\rho_1}}\frac{K_{\lambda_1}}{2}-K_{\eta_1}\frac{K_{\lambda_1}}{2}+K_{\rho_1}K_{\lambda_1}+K_{c_{p1}}\frac{K_{\lambda_1}}{2}\right.$$
$$\left.-\frac{K_{\lambda_1}^2}{2}\right)+\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12}\right)(6K_{\lambda_1}^2-2K_{\lambda_2})\Bigg\}$$

$$\begin{aligned}
&+\varepsilon^2 A^2 \frac{Gr^2\,Pr^2}{362\,880}\left\{\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{12}\right)\left(-10K_{\rho_1}K_{\lambda_1}\right.\right.\\
&-10\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}+10K_{\eta_1}K_{\lambda_1}+11K_{\lambda_1}^2-15K_{c_{p1}}K_{\lambda_1}\\
&+3K_{\eta_1}^2-4\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}-4K_{\rho_1}K_{\eta_1}-K_{\eta_2}+5K_{\rho_2}\\
&+6\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}+6K_{\rho_1}K_{c_{p1}}-6K_{c_{p1}}K_{\eta_1}+2K_{c_{p2}}\\
&\left.-\frac{K_{\lambda_2}}{2}+2K_{c_{p1}}^2+K_{\rho_1}^2+\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2\right)-\left(\frac{x^2}{2}-\frac{x^3}{3}-\frac{x}{6}\right)\\
&\times\left(2K_{\rho_1}K_{\lambda_1}+2\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-2K_{\eta_1}K_{\lambda_1}-\frac{5}{2}K_{\lambda_1}^2\right.\\
&\left.\left.+3K_{c_{p1}}K_{\lambda_1}\right)-\left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12}\right)(9K_{\lambda_1}^2-3K_{\lambda_2})\right\}.
\end{aligned} \tag{33}$$

Knowing the velocity and temperature field in the core region, the Nusselt number can be determined. Since the lower and upper walls are adiabatic, the flow of enthalpy and heat must be constant at every cross-section of the cavity, hence

$$Nu = \int_0^1 \left(\lambda \frac{\partial \theta}{\partial x} - Gr\,Pr\,\rho u c_p \theta\right) \mathrm{d}y. \tag{34}$$

With $T_R = (T_C + T_H)/2$ one obtains for the Nusselt number

$$\begin{aligned}
Nu = {} & 1+\varepsilon^2\frac{K_{\lambda_2}}{24}+\frac{A^2\,Gr^2\,Pr^2}{362\,880}\\
&\times\left[1+\varepsilon^2\left(-\frac{1}{2}K_{\rho_1}K_{\lambda_1}-\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\right.\right.\\
&+\frac{1}{2}K_{\eta_1}K_{\lambda_1}+\frac{1}{2}K_{\lambda_1}^2-\frac{3}{4}K_{c_{p1}}K_{\lambda_1}\\
&+\frac{1}{4}K_{\eta_1}^2-\frac{1}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}-\frac{1}{3}K_{\rho_1}K_{\eta_1}\\
&-\frac{K_{\eta_2}}{12}+\frac{5}{12}K_{\rho_2}+\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}\\
&+\frac{1}{2}K_{\rho_1}K_{c_{p1}}-\frac{1}{2}K_{c_{p1}}K_{\eta_1}+\frac{K_{c_{p2}}}{6}\\
&\left.\left.-\frac{K_{\lambda_2}}{24}+\frac{1}{6}K_{c_{p1}}^2+\frac{K_{\rho_1}^2}{12}+\frac{1}{12}\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2\right)\right].
\end{aligned} \tag{35}$$

In addition to this, we also present the solution for the Nusselt number for $T_R = T_C$ instead of $T_R = (T_C + T_H)/2$

$$\begin{aligned}
Nu = {} & 1+\varepsilon\frac{K_{\lambda_1}}{2}+\varepsilon^2\frac{K_{\lambda_2}}{6}+\frac{A^2\,Gr^2\,Pr^2}{362\,880}\\
&\times\left[1+\varepsilon\left(K_{\rho_1}+\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-\frac{1}{2}K_{\lambda_1}\right.\right.\\
&\left.+\frac{3}{2}K_{c_{p1}}\right)+\varepsilon^2\left(\frac{13}{4}K_{\lambda_1}^2-\frac{2}{3}K_{\lambda_2}-\frac{13}{6}K_{\rho_1}K_{\lambda_1}\right.\\
&-\frac{13}{6}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}+\frac{13}{6}K_{\eta_1}K_{\lambda_1}-\frac{5}{3}K_{\eta_1}K_{\lambda_1}\\
&-\frac{5}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}+\frac{4}{3}K_{\eta_1}^2-\frac{K_{\eta_2}}{2}+\frac{13}{6}K_{\rho_2}\\
&-\frac{5}{3}K_{c_{p1}}K_{\lambda_1}+\frac{7}{3}K_{\rho_1}K_{c_{p1}}+\frac{7}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}\\
&-\frac{7}{3}K_{\eta_1}K_{c_{p1}}+\frac{5}{6}K_{c_{p2}}+\frac{K_{\rho_1}^2}{3}+\frac{1}{3}\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2\\
&\left.\left.+\frac{3}{2}k_{c_{p1}}^2\right)\right].
\end{aligned} \tag{36}$$

The terms of $O(\varepsilon^0)$ in equations (35) and (36) are identical with the Boussinesq solution (equations (27)–(30)) which has already been given by Cormack *et al.* The terms of $O(\varepsilon^1)$ and $O(\varepsilon^2)$ are corrections due to the temperature-dependent properties. It is interesting to note that no terms of $O(\varepsilon^1)$ show up, if the arithmetic mean is chosen as the reference temperature. The solutions for the velocity and temperature field in the case $T_R = T_C$ are given in the Appendix.

The results are sometimes presented by using the property-ratio method, see Herwig [7]. But this can only be done, in case the corrections due to the variable properties are of $O(\varepsilon^1)$, because only in this case are the exponents of the property ratios independent of the properties K_{α_i} of the second kind. Because there are no terms of $O(\varepsilon^1)$ in the relation for the Nusselt number in the case $T_R = (T_H + T_C)/2$, this solution cannot be presented by the property ratio method. However, one obtains in the case $T_R = T_C$

$$\frac{Nu}{Nu_B}\left[\frac{\eta_H\rho_H\beta_C}{\eta_C\beta_H\rho_C}\right]^{n_1}\left[\frac{\lambda_H}{\lambda_C}\right]^{n_2}\left[\frac{\rho_H}{\rho_C}\right]^{n_3}\left[\frac{c_{pH}}{c_{pC}}\right]^{n_4} \tag{37}$$

with

$$\frac{\rho_H\beta_C}{\beta_H\rho_C} = 1+\varepsilon\frac{2K_{\rho_1}^2-K_{\rho_2}/K_{\rho_1}}{K_{\rho_1}}$$

$$n_1 = -\frac{A^2\,Gr^2\,Pr^2}{362\,880+A^2\,Gr^2\,Pr^2}$$

$$n_2 = \frac{181\,440-Gr^2\,Pr^2\,A^2/2}{362\,880+A^2\,Gr^2\,Pr^2}$$

$$n_3 = 3n_1$$

$$n_4 = \tfrac{3}{2}n_1.$$

Subscript B means 'Boussinesq approximation'.

4. DISCUSSION

The results of the study are presented in Figs. 2 and 3 for a shallow cavity with aspect ratio $A = 0.02$. The graphs show the relative deviations of the Nusselt number from that obtained by applying the Boussinesq approximation, $(Nu-Nu_B)/Nu_B$, vs the parameter $\varepsilon = (T_H-T_C)/T_R$. The value $\varepsilon = 0.1$ means a temperature difference of approximately 30 K between the hot and cold wall, if the arithmetic mean is close to the ambient temperature. Because the Grashof number is proportional to the temperature difference (T_H-T_C) if the cavity aspect ratio A is kept constant, the Grashof number can be used instead of ε in Figs. 2 and 3.

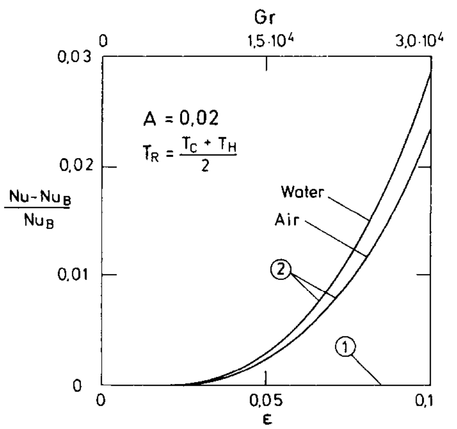

FIG. 2. Relative deviation between the Nusselt number for constant and variable properties vs the parameter ε in the case $T_R = (T_C+T_H)/2$ and for $A = 0.02$: ①, linear theory; ②, quadratic theory.

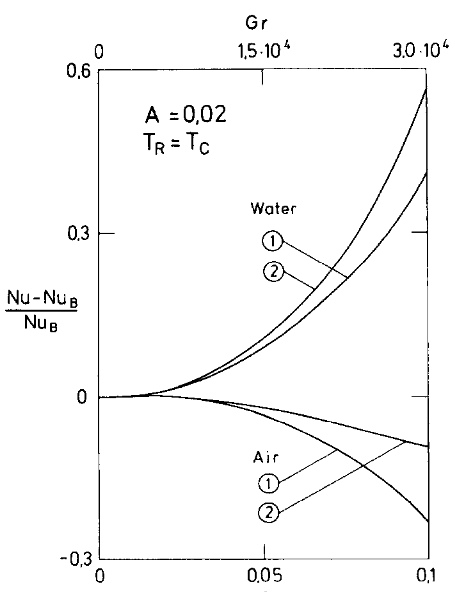

FIG. 3. Relative deviation between the Nusselt number for constant and variable properties vs the parameter ε in the case $T_R = T_C$ and for $A = 0.02$: ①, linear theory; ②, quadratic theory.

Figure 2 presents the results for the case $T_R = (T_C+T_H)/2$. It is interesting to note that the solution obtained is correct within the framework of the linear theory. The Nusselt numbers for water and air are slightly increased, if the quadratic terms are taken into account. But, this increase is rather small and is approximately 2.5% for $Gr = 3\times10^4$ or $\varepsilon = 0.10$, respectively.

Figure 3 shows the results for the case $T_R = T_C$. The curves marked ① represent the solution of the linear theory and those marked ② represent the quadratic theory. With $T_R = T_C$, the effect of the variable properties is opposite for water and air, i.e. the Nusselt number for water (linear theory) can be increased up to 42% and that one for air can be decreased down to 24%. If the quadratic terms are added, the deviation of the Nusselt number for water is further increased (up to 57%), that one for air, on the contrary, is slightly decreased.

Summing up, the study presented shows that the arithmetic mean temperature is a reasonable choice for the reference temperature in the present problem. Using this reference temperature, the Boussinesq solution is exact compared to the linear theory and is still very good (maximum deviations less than 2.5%) with the quadratic terms being added. Contrary to this, the deviation between the quadratic theory and the Boussinesq solution is rather high (up to about 50%) and, in addition, oppositely for water and air, in case the temperature of the cold side wall is chosen as the reference temperature. Hence, for cavity flow problems it is suggested that the arithmetic mean temperature is used as reference temperature for the variable properties.

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APPENDIX A: GOVERNING EQUATION OF $O(\varepsilon^1)$

Equation of continuity

$$A\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+AK_{\rho_1}\frac{\partial\Theta_0}{\partial x}u_0+K_{\rho_1}\frac{\partial\Theta_0}{\partial y}v_0=0. \quad \text{(A1)}$$

Equation of motion

$$\begin{aligned}Gr\Bigg[&A^2u_0\frac{\partial u_1}{\partial x}+A^2u_1\frac{\partial u_0}{\partial x}+A^2K_{\rho_1}\Theta_0u_0\frac{\partial u_0}{\partial x}\\&+Av_0\frac{\partial u_1}{\partial y}+Av_1\frac{\partial U_0}{\partial y}+AK_{\rho_1}\Theta_0v_0\frac{\partial u_0}{\partial y}\Bigg]\\&=-Gr\frac{\partial p_1}{\partial y}+\frac{4}{3}A^2\frac{\partial^2u_1}{\partial x^2}-\frac{2}{3}A\frac{\partial^2v_1}{\partial x\,\partial y}+\frac{\partial^2u_1}{\partial y^2}\\&+A\frac{\partial^2v_1}{\partial x\,\partial y}+\frac{4}{3}A^2K_{\eta_1}\frac{\partial\Theta_0}{\partial x}\frac{\partial u_0}{\partial x}+\frac{4}{3}A^2K_{\eta_1}\Theta_0\frac{\partial^2u_0}{\partial x^2}\\&-\frac{2}{3}AK_{\eta_1}\frac{\partial\Theta_0}{\partial x}\frac{\partial v_0}{\partial y}-\frac{2}{3}AK_{\eta_1}\Theta_0\frac{\partial^2v_0}{\partial x\,\partial y}+K_{\eta_1}\frac{\partial\Theta_0}{\partial y}\frac{\partial u_0}{\partial y}\\&+K_{\eta_1}\Theta_0\frac{\partial^2u_0}{\partial y^2}+AK_{\eta_1}\frac{\partial\Theta_0}{\partial y}\frac{\partial v_0}{\partial x}+AK_{\eta_1}\Theta_0\frac{\partial^2v_0}{\partial x\,\partial y} \quad \text{(A2)}\end{aligned}$$

$$\begin{aligned}Gr\Bigg[&A^2u_0\frac{\partial v_1}{\partial x}+A^2u_1\frac{\partial v_0}{\partial x}+A^2K_{\rho_1}\Theta_0u_0\frac{\partial v_0}{\partial x}\\&+Av_0\frac{\partial v_1}{\partial y}+Av_1\frac{\partial v_0}{\partial y}+AK_{\rho_1}\Theta_0v_0\frac{\partial v_0}{\partial y}\Bigg]\\&=-Gr\frac{\partial p_1}{\partial y}+\frac{4}{3}\frac{\partial^2v_1}{\partial y^2}-\frac{2}{3}A\frac{\partial^2u_1}{\partial x\,\partial y}+A\frac{\partial^2u_1}{\partial x\,\partial x}+A^2\frac{\partial^2v_1}{\partial x^2}\\&+\frac{4}{3}K_{\eta_1}\frac{\partial\Theta_0}{\partial y}\frac{\partial v_0}{\partial y}+\frac{4}{3}K_{\eta_1}\Theta_0\frac{\partial^2v_0}{\partial y^2}-\frac{2}{3}AK_{\eta_1}\frac{\partial\Theta_0}{\partial y}\frac{\partial u_0}{\partial x}\\&-\frac{2}{3}AK_{\eta_1}\Theta_0\frac{\partial^2u_0}{\partial x\,\partial y}+AK_{\eta_1}\frac{\partial\Theta_0}{\partial x}\frac{\partial u_0}{\partial y}+AK_{\eta_1}\Theta_0\frac{\partial^2u_0}{\partial x\,\partial y}\\&+A^2K_{\eta_1}\frac{\partial\Theta_0}{\partial x}\frac{\partial v_0}{\partial x}+A^2K_{\eta_1}\Theta_0\frac{\partial^2v_0}{\partial x_2}+\Theta_1+\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}\Theta_0^2. \quad \text{(A3)}\end{aligned}$$

Equation of energy

$$\begin{aligned}Gr\,Pr\Bigg[&A^2u_0\frac{\partial\Theta_1}{\partial x}+A^2u_1\frac{\partial\Theta_0}{\partial x}+A^2K_{\rho_1}\Theta_0u_0\frac{\partial\Theta_0}{\partial x}\\&+A^2K_{c_{p1}}\Theta_0u_0\frac{\partial\Theta_0}{\partial x}+Av_0\frac{\partial\Theta_1}{\partial y}+Av_1\frac{\partial\Theta_0}{\partial y}\\&+AK_{\rho_1}\Theta_0v_0\frac{\partial\Theta_0}{\partial y}+AK_{c_{p1}}\Theta_0v_0\frac{\partial\Theta_0}{\partial y}\Bigg]\\&=A^2\frac{\partial^2\Theta_1}{\partial x_2}+A^2K_{\lambda_1}\left(\frac{\partial\Theta_0}{\partial x}\right)^2+A^2K_{\lambda_1}\Theta_0\frac{\partial^2\Theta_0}{\partial x^2}\\&+\frac{\partial^2\Theta_1}{\partial y^2}+K_{\lambda_1}\left(\frac{\partial\Theta_0}{\partial y}\right)^2+K_{\lambda_1}\Theta_0\frac{\partial^2\Theta_0}{\partial y^2}. \quad \text{(A4)}\end{aligned}$$

APPENDIX B: VELOCITY AND TEMPERATURE PROFILE IN THE CORE REGION FOR $T_R = T_C$

$$\begin{aligned}u=\frac{1}{24}(4y^3-6y^2+2y)\Bigg[&1+\varepsilon\left[\frac{K_{\lambda_1}}{2}+x\left\{\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-k_{\lambda_1}\right\}\right]\\&+\varepsilon^2\Bigg[\frac{K_{\lambda_2}}{6}+x\left\{\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-K_{\eta_1}K_{\lambda_1}-K_{\lambda_1}^2\right\}\\&+x^2\Bigg\{\frac{3}{2}K_{\eta_1}K_{\lambda_1}-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}+K_{\eta_1}^2\\&-\frac{K_{\eta_2}}{2}+\frac{3}{2}K_{\lambda_1}^2-\frac{K_{\lambda_2}}{2}-\frac{3}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\Bigg\}\Bigg]\Bigg] \quad \text{(B1)}\end{aligned}$$

$$\begin{aligned}\Theta=x&+\varepsilon\frac{K_{\lambda_1}}{2}(x-x^2)+\varepsilon^2\left[\frac{K_{\lambda_1}^2}{2}(x^3-x^2)+\frac{K_{\lambda_2}}{6}(x-x^3)\right]\\&+A^2\frac{Gr\,Pr}{24}\left(\frac{y^5}{5}-\frac{y^4}{2}+\frac{y^3}{3}-\frac{1}{60}\right)\\&+\varepsilon A^2\Bigg[\frac{Gr\,Pr}{24}\left(\frac{y^5}{5}-\frac{y^4}{2}+\frac{y^3}{3}-\frac{1}{60}\right)\\&\times\left(K_{\lambda_1}+x\left(K_{\rho_1}+\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-3K_{\lambda_1}+K_{c_{p1}}\right)\right)\\&-\frac{Gr^2\,Pr^2}{362\,880}\left(K_{\rho_1}+\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-2K_{\lambda_1}+\frac{3}{2}K_{c_{p1}}\right)\\&\times(x^2-x)\Bigg]+\varepsilon^2A^2\Bigg[\frac{Gr\,Pr}{24}\left(\frac{y^5}{5}-\frac{y^4}{2}+\frac{y^3}{3}-\frac{1}{60}\right)\\&\times\Bigg\{x^2\Bigg(\frac{15}{2}K_{\lambda_1}^2-\frac{3}{2}K_{\lambda_2}-\frac{7}{2}K_{\rho_1}K_{\lambda_1}+\frac{7}{2}K_{\eta_1}K_{\lambda_1}-K_{\eta_1}K_{\rho_1}\\&-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}+K_{\eta_1}^2-\frac{K_{\eta_2}}{2}+\frac{3}{2}K_{\rho_2}-\frac{7}{2}K_{c_{p1}}K_{\lambda_1}+K_{\rho_1}K_{c_{p1}}\\&+\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}-K_{\eta_1}K_{c_{p1}}+\frac{K_{c_{p2}}}{2}\Bigg)+x\Bigg(-\frac{9}{2}K_{\lambda_1}^2+\frac{3}{2}K_{\rho_1}K_{\lambda_1}\\&+\frac{3}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-\frac{3}{2}K_{\eta_1}K_{\lambda_1}+\frac{3}{2}K_{c_{p1}}K_{\lambda_1}\Bigg)+\frac{K_{\lambda_1}^2}{4}+\frac{K_{\lambda_2}}{3}\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{Gr^2\,Pr^2}{362\,880}\Bigg\{(x^3-x)\Bigg(4K_{\rho_1}K_{\lambda_1}+4\frac{K_{\rho_2}}{K_{\rho_2}}K_{\lambda_1}-4K_{\eta_1}K_{\lambda_1}\\
&-6K_{\lambda_1}^2+\frac{20}{3}K_{c_{p1}}K_{\lambda_1}+\frac{2}{3}K_{\lambda_2}+\frac{4}{3}K_{\eta_1}K_{\rho_1}+\frac{4}{3}K_{\eta_1}\frac{K_{\rho_2}}{K_{\rho_1}}\\
&-K_{\eta_1}^2+\frac{K_{\eta_2}}{6}-\frac{5}{3}K_{\rho_2}-2K_{\rho_1}K_{c_{p1}}-2\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}\\
&+2K_{\eta_1}K_{c_{p1}}-\frac{2}{3}K_{c_{p2}}-\frac{1}{3}K_{\rho_1}^2-\frac{1}{3}\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2-\frac{2}{3}K_{c_{p1}}\Bigg)\\
&+(x^2-x)\Bigg(6K_{\lambda_1}^2-3K_{\rho_1}K_{\lambda_1}-3\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\\
&+3K_{\eta_1}K_{\lambda_1}-\frac{9}{2}K_{c_{p1}}K_{\lambda_1}\Bigg)\Bigg\}\Bigg].
\end{aligned}
\tag{B2}
$$

CONVECTION NATURELLE DANS UNE CAVITE PEU PROFONDE—1. FLUIDE NEWTONIEN

Résumé—On étude analytiquement l'effet des propriétés variables sur la convection naturelle dans une cavité peu profonde avec parois d'extrémité chauffées différemment. Par rapport aux résultats de la *théorie linéaire*, l'approximation de Boussinesq usuellement appliquée est exacte si les propriétés du fluide sont prises à la température moyenne arithmétique entre celles des parois chaude et froide. Pour le domaine de variation considéré la déviation entre les résultats pour la *théorie quadratique* et ceux pour l'approximation de Boussinesq est inférieure à 2,5% environ. On montre que la moyenne arithmétique est un choix raisonnable. Avec cette température de référence, l'approximation de Boussinesq conduit à des résultats suffisamment précis.

FREIE KONVEKTION IN EINEM FLACHEN BEHÄLTER MIT VARIABLEN STOFFWERTEN—1. NEWTONSCHES FLUID

Zusammenfassung—Der Einfluß variabler Stoffwerte aud die freie Konvektion in einem flachen Behälter mit unterschiedlich beheizten Stirnflächen wird analytisch untersucht. Wie die Ergebnisse zeigen, liefert die Boussinesq-Approximation im Vergleich mit der *linearen Theorie* eine exakte Lösung, wenn die arithmetische Mitteltemperatur zwischen heißer und kalter Stirnfläche als Referenztemperatur verwendet wird. Im untersuchten Parameter-Bereich weicht die Boussinesq-Lösung um maximal 2.5% von der "exakten Lösung", d.h. von der *quadratischen Theorie* ad. Damit its gezeigt, daß die arithmetische Mittletemperatur eine sinnvolle Bezugstemperatur für die Stoffwerte ist. Mit dieser Bezugstemperatur liefert die Boussinesq-Approximation hinreichend genaue Ergebnisse.

ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ С УЧЕТОМ ПЕРЕМЕННОСТИ СВОЙСТВ В МЕЛКОЙ ПОЛОСТИ—1. НЬЮТОНОВСКАЯ ЖИДКОСТЬ

Аннотация—Проведено аналитическое исследование влияния переменных свойств на естественную конвекцию в мелкой полости, торцевые стенки которой поддерживаются при разной температуре. Результаты для линейной температурной зависимости точно совпадают с обычно применяемым приближением Буссинеска, если свойства жидкости берутся при значении температуры, которое равно средне-арифметической температуре. В рассматриваемом диапазоне изменения параметров расхождения между результатами, получаемыми для температурной зависимости, и приближением Буссинеска составляет менее 2,5%. Показано, что приближение Буссинеска при выборе среднеарифметической температуры в качестве характерной температуры дает достаточно точные результаты.